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STEADY MOTIONS AND INTEGRAL MANIFOLDS OF SYSTEMS WITH QUADRATIC INTEGRALS*

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An investigation is made of conservative systems with an additional integral of motion which is quadratic in the velocity. A method which takes into account the specific features of the mechanical problems is proposed to describe steady motions and integral surfaces in phase space. As an example, a non-holonomic problem, involving the motion of a rigid body carrying a gyroscope is considered.

Topological analysis of mechanical systems with known integrals $F_1, \ldots, F_k$ aims at describing the surfaces in phase space defined by fixed values of the integrals and studying the bifurcations of these surfaces /1/. The bifurcation points are defined by a dependence condition involving the integrals, $\Sigma\lambda_i dF_i = 0$ (λ_i (where λ_i are Lagrange multipliers), or $dF_\lambda = 0$, where $F_\lambda = \Sigma\lambda_i F_i$ is a pencil of integrals with constant coefficients λ_i. The condition $dF_\lambda = 0$ is invariant /2/, i.e., it holds along the whole trajectory of the system emanating from a critical point of the pencil F_λ. The motion in this case is said to be steady. Such motions have been studied by numerous authors, e.g., /3-7/. In the typical case they form families parametrized by the values of the constants λ_i.

Thus, topological analysis involves the description of steady motions. When the integrals (other than the entry) are linear in the velocity, both problems can be tackled by means of reduced potentials /1, 8/. In this paper, consideration will be given to functions which play an analogous role for a conservative system with an additional integral which is a quadratic function of the velocity.

1. Let M be a configurational manifold with Riemannian form $\langle\cdot,\cdot\rangle$. In order to include the non-holonomic case, our phase space will be an m-dimensional subbundle $T'M$ of the tangent bundle TM: at every point $x \in M$ the fibre $T_x'M$ of this subbundle is the space of velocities allowable by the constraints (in the holonomic case $T'M = TM$). Assume that the integrals are

$$H(\mathbf{v}) = {}^1/_2\langle \mathbf{v}, \mathbf{v}\rangle + V(x),\ F(\mathbf{v}) = {}^1/_2\langle \Gamma\mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{a}, \mathbf{v}\rangle + W(x)$$

where $\mathbf{v} \in T'M$ is the velocity vector at the point $x \in M$, V and W are functions of the positional variables, Γ is a symmetric linear bundle operator, and $\mathbf{a}$ is a vector field on M. We may assume that Γ acts from $T'M$ to $T'M$ and that $\mathbf{a} \in T'M$; otherwise we replace them respectively by $Pr \circ \Gamma$ and $Pr(\mathbf{a})$, where Pr is the bundle operator of orthogonal projection

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onto $T'M$.

For convenience, we shall temporarily assume that over every point x of the operator Γ lie m distinct eigenvalues $\mu_1(x) < \ldots < \mu_m(x)$, and $\mathbf{a}_i(x) \neq 0$, $i = 1, \ldots, m$, where $\mathbf{a}_i$ is the component of $\mathbf{a}$ in the direction of an eigenvector of Γ, $\Gamma\mathbf{a}_i = \mu_i\mathbf{a}_i$, $\Sigma\mathbf{a}_i = \mathbf{a}$.

Let us find the critical points of the pencil of integrals F_λ:

$$F_\lambda = \lambda H + F = {}^1/_2\langle(\Gamma + \lambda E)\,\mathbf{v},\ \mathbf{v}\rangle + \langle\mathbf{a},\ \mathbf{v}\rangle + \lambda V + W. \tag{1.1}$$

Suppose that the partial derivatives of F_λ with respect to the velocities vanish at a point $\mathbf{v} \in T'M$, i.e., $(\Gamma + \lambda E)\,\mathbf{v} + \mathbf{a} = 0$. Vectors $\mathbf{v}$ satisfying this condition are called critical vectors for the given value of λ. Over every point x there is a critical vector, which is uniquely defined if $\lambda \neq -\mu_i(x)$; we denote it by

$$\mathbf{v}_\lambda = -(\Gamma + \lambda E)^{-1}\mathbf{a} \tag{1.2}$$

There are no critical vectors over a point x for which $\lambda = -\mu_i(x)$, since $\mathbf{a}_i \neq 0$. The critical points of an integral F_λ are singled out of the set of vectors (1.2) by the condition that the differential of the function

$$\Phi_\lambda(x) = F_\lambda(\mathbf{v}_\lambda) = -{}^1/_2\langle(\Gamma + \lambda E)^{-1}\mathbf{a},\ \mathbf{a}\rangle + \lambda V + W \tag{1.3}$$

which is defined throughout M except at points where $-\mu_i(x) = \lambda$, should vanish. Consequently, the critical points $\mathbf{v}$ of the integrals H, F for a given λ are defined by the condition

$$\mathbf{v} = \mathbf{v}_\lambda(x),\ d\Phi_\lambda(x) = 0 \tag{1.4}$$

For every λ there are steady motions through the critical points of the function Φ_λ, with velocity $\mathbf{v}_\lambda$.

Formally speaking, the condition $dH = 0$ corresponds to $\lambda = \infty$. Instead of (1.4), we obtain $\mathbf{v} = 0$, $dV = 0$, which defines equilibrium points.

We will establish certain relations for Φ_λ and $\mathbf{v}_\lambda$.

Proposition 1. For every λ, the function Φ_λ is invariant with respect to the field $\mathbf{v}_\lambda$.

Proof. In each fibre $T_x'M$ we apply the transformation $\mathbf{v} \to \mathbf{w} = (\Gamma + \lambda E)\,\mathbf{v} + \mathbf{a}$. It then follows from (1.1) and (1.3) that

$$F_\lambda = {}^1/_2\langle(\Gamma + \lambda E)^{-1}\,\mathbf{w},\ \mathbf{w}\rangle + \Phi_\lambda$$

Along an arbitrary trajectory, we have

$$0 = \frac{dF_\lambda}{dt} = {}^1/_2\left\langle\frac{d}{dt}(\Gamma + \lambda E)^{-1}\,\mathbf{w},\ \mathbf{w}\right\rangle + \left\langle(\Gamma + \lambda E)^{-1}\mathbf{w},\ \frac{d\mathbf{w}}{dt}\right\rangle + \frac{d\Phi_\lambda}{dt}$$

and if the initial velocity is $\mathbf{v}_\lambda$, then at $t = 0$ we have $\mathbf{w} = \mathbf{w}(\mathbf{v}_\lambda) = 0$, the derivative of Φ_λ equals $\mathbf{v}_\lambda(\Phi_\lambda)$, and so $\mathbf{v}_\lambda(\Phi_\lambda) = 0$, as required.

Let us consider Φ_λ as a function $\Phi(\lambda, x) = \Phi_\lambda(x)$, defined everywhere on $R \times M$ except for the surfaces of discontinuity $\{\mu_i(x) + \lambda = 0\}$. It follows from (1.1)-(1.3) that

$$H(\mathbf{v}_\lambda) = {}^1/_2\langle(\Gamma + \lambda E)^{-2}\mathbf{a},\ \mathbf{a}\rangle + V = \partial\Phi/\partial\lambda \tag{1.5}$$
$$F(\mathbf{v}_\lambda) = \Phi - \lambda H(\mathbf{v}_\lambda) = \Phi - \lambda\partial\Phi/\partial\lambda$$

2. The surfaces $I_{hf} \subset T'M$ corresponding to fixed values of the integrals $H = h$, $F = f$ are the preimages of the pairs (h, f) under the integral mapping $H \times F : T'M \to R^2$. Their topological type is invariant to small variations of a point $(h, f) \in R^2$ in the general position but it changes when (h, f) passes through a bifurcation set $\Sigma \subset R^2$ which includes pairs of critical values of integrals (and is exhausted by them if all the I_{hf} are compact).

In the case in hand the critical points of the integral mapping are determined by the critical points of the functions Φ_λ. By Proposition 1, for each λ the latter form a set which is invariant under $\mathbf{v}_\lambda$. In the typical case, varying λ gives a smooth family of diffeomorphic sets of critical points. Let $h(\lambda)$, $f(\lambda)$ be critcal values of the integrals determined according to (1.4) by the critical points of Φ_λ in one of these families. Then the curve $(h(\lambda),\ f(\lambda))$ parametrized by λ occurs in a bifurcation set.

Proposition 2. The following equality holds on the above-mentioned bifurcation curve:

$$df/dh = -\lambda$$

Proof. In each critical set of Φ_λ, choose a point $x(\lambda)$ so as to obtain a smooth curve in M. By the definition of the quantities $h(\lambda)$, $f(\lambda)$ and by (1.3), $\lambda h(\lambda) + f(\lambda) = \Phi_\lambda(x(\lambda))$. Differentiating with respect to λ and using the fact that $d\Phi_\lambda = 0$ at $x(\lambda)$, we deduce from (1.5) that $\lambda h' + f' = 0$ which is equivalent to the required assertion.

3. We will not consider the projection $\pi: I_{hf} \to M$ of the integral surface onto the configuration manifold. The mechanical meaning of the mapping π was pointed out, for example, by Orekhov* (*Orekhov V.I., Geometrical and topological analysis of integrals of motion in problems of analytical mechanics. Candidate Dissertation, Moscow Univ., Moscow, 1970) and in greater detail in /9/: the image $\pi(I_{hf}) = M_{hf}$ is the domain of possible motion (DPM) for the given values of the integrals; the preimage $\pi^{-1}(x)$, i.e., the section $I_{hf} \cap T_x'M$, is the set of possible velocities at x. The set of critical images of π is called the generalized boundary of the DPM /9/; over this set the sections $\pi^{-1}(x)$ bifurcate.

DPMs and their generalized boundaries have been described for some integrable problems of rigid-body dynamics with inhomogeneous quadratic integrals /9/. Our aim is to present a general approach to the description of DPMs M_{hf} and their generalized boundaries δM_{hf} in terms of the functions $\Phi_\lambda(x)$ or $\Phi(\lambda, x)$.

Let $\Phi_h(\lambda, x) = \Phi(\lambda, x) - \lambda h$ be a new function in $R \times M$. Let us consider the level surface $S = \{\Phi_h = f\}$ and its projection $S \to M$. The sections $S \cap \{\lambda = \text{const}\}$ project into level surfaces of the functions Φ_λ on M, which we denote by $P(\lambda)$; $P(\lambda) = \{\Phi_\lambda(x) - \lambda h = f\}$.

Proposition 3. The generalized boundary δM_{hf} is the set of critical images of the projection $S \to M$ or, what is the same, the enveloping family of the surfaces $P(\lambda)$.

Proof. A point $x \in M$ lies in δM_{hf} provided that $\pi^{-1}(x)$ contains a critical point of the mapping π, i.e., a critical vector $\mathbf{v}_\lambda \in I_{hf}$. Then $H(\mathbf{v}_\lambda) = h$, $F(\mathbf{v}_\lambda) = f$ and, by (1.5),

$$\partial\Phi/\partial\lambda - h = 0, \quad \Phi(\lambda, x) - \lambda h = f$$

which proves the assertion.

Expanding a in a series of eigenfunctions of Γ, we can write

$$\Phi_h(x, \lambda) = -{}^1/_2 \sum_{i=1}^{m} \frac{\mathbf{a}_i^2(x)}{\mu_i(x) + \lambda} + \lambda(V(x) - h) + W(x) \tag{3.1}$$

whence we see that S splits into components S_j, $j = 0, 1, \ldots, m$, separated by the surfaces of discontinuity $\{\mu_i(x) + \lambda = 0\}$ of the function (3.1). Accordingly, in each surface $P(\lambda) \subset M$ we define components

$$P_j(\lambda) = P(\lambda) \cap \{-\mu_j(x) < \lambda < -\mu_{j-1}(x)\}, \quad j = 2, \ldots, m$$
$$P_1(\lambda) = P(\lambda) \cap \{-\mu_1(x) < \lambda\}, \quad P_0(\lambda) = P(\lambda) \cap \{-\mu_m(x) > \lambda\}$$

For each λ, we also consider the domains

$$C_1(\lambda) = \{\Phi_\lambda - \lambda h < f\} \cap \{-\mu_1(x) < \lambda\} \subset M$$
$$C_0(\lambda) = \{\Phi_\lambda - \lambda h > f\} \cap \{-\mu_m(x) > \lambda\} \subset M$$

Let F_x be the restriction of the integral F to the sphere $\{\mathbf{v} \in T_x' M : H(\mathbf{v}) = h\}$ over x. The critical values of F_x are $F(\mathbf{v}_\lambda)$, and by (1.5) they coincide with the critical values of Φ_h as functions of λ for fixed x. Hence it follows that as x is varied the level sets $\{F_x = f\}$ and $\{\Phi_h = f\}$ bifurcate simultaneously, and thus the topological type of the set of possible velocities $\pi^{-1}(x) = \{F_x = f\}$ is uniquely defined by the distribution of roots of the equation $\Phi_h = f$ on the axis $\{\lambda\} \times x$. For each root there is a point over x on the surface S_j, as well as a curve $P_j(\lambda)$ passing through x. Omitting the detailed proofs, we present the results implied by this correspondence.

Proposition 4. The DPMs M_{hf} are described by the following equalities:

$$M \setminus M_{hf} = \bigcup_\lambda (C_1(\lambda) \cup C_0(\lambda))$$
$$M \setminus \operatorname{Int} M_{hf} = \bigcup_\lambda (P_1(\lambda) \cup P_0(\lambda))$$

Let D_1 be the set of points $x \in M$ through which there is exactly one surface from each family P_j, $j = 2, \ldots, m$, and no surface from P_1, P_0; let D_i, $i = 2, \ldots, m$, be the set of points through each of which there pass three surfaces from the family P_i, one surface from each of the families P_j, $j = 2, \ldots, i-1, i+1, \ldots, m$, and no surface from P_1, P_0.

Proposition 5. The domain $M_{hf} \setminus \delta M_{hf}$ is the union of all the domains $D_{1,\ldots,m}$. Over all points of each connected component of D_i the sets of possible velocities are diffeomorphic to one another.

4. We will now weaken our assumption about the eigenvalues of Γ and eigencomponents of a. We assume the existence in M of surfaces $\{\mu_i(x) = \mu_{j+1}(x)\}$ and $\{\mathbf{a}_i(x) = 0\}$. Over a point $x \in \{\mathbf{a}_i = 0\}$ the critical vector $\mathbf{v}$ for $\lambda = -\mu_i(x)$ is not uniquely defined by the condition $(\Gamma + \lambda E)\mathbf{v} + \mathbf{a} = 0$. Suppose that one of these vectors is a critical point of the integrals. Let us consider the steady motion starting at that point. If the trajectory intersects the set $\{-\mu_i(x) = \lambda\}$ at isolated points, then condition (1.4) will hold at all

other points of the trajectory, which is thus the closure of the set of critical points of Φ_λ. Otherwise, we obtain a motion confined to the points of a surface $\{-\mu_i(x) = \lambda\} \cap \{\mathbf{a}_i(x) = 0\}$ of lower dimension; we will omit the detailed analysis of this case.

Considering the projection $\pi: I_{hf} \to M$, we observe that its critical preimages may include vectors $\mathbf{v} \neq \mathbf{v}_\lambda$ over points $x \in \{-\mu_i = \lambda\} \cap \{\mathbf{a}_i = 0\}$. Proposition 3 may be refined as follows: δM_{hf} is the closure of the set of critical images of the projection $S \to M$ and the enveloping family of the surfaces $P(\lambda)$. In Proposition 4, the sets $\bigcup P_0(\lambda)$, $\bigcup P_1(\lambda)$ are replaced by their closures. In the description of the domains D_i figuring in Proposition 5, one must consider, along with $P_j(\lambda)$, the surfaces $\{\mu_j(x) = \mu_{j+1}(x)\}$ as well.

5. As an example, let us consider the motion of a rigid body, fixed at its centre of mass, carrying a rigidly fixed gyroscope with constant angular momentum $\mathbf{k}$. Suppose that the system is subject to a non-holonomic constraint $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0$, where $\boldsymbol{\omega}$ is the angular velocity and $\boldsymbol{\gamma}$ a unit vector along a fixed axis. The problem has integrals /10/

$$H = {}^1/_2 (J\boldsymbol{\omega}, \boldsymbol{\omega}),\ F = {}^1/_2 [\mathbf{K}^2 - (\mathbf{K}, \boldsymbol{\gamma})^2]$$

Here J is the inertia tensor, whose principal values are denoted by $J_1 > J_2 > J_3$, $\mathbf{K} = J\boldsymbol{\omega} + \mathbf{k}$ is the kinetic moment of the system. In this case

$$\Gamma\boldsymbol{\omega} = J\boldsymbol{\omega} - (J\boldsymbol{\omega}, \boldsymbol{\gamma})\boldsymbol{\gamma},\quad \mathbf{a} = \mathbf{k} - (\mathbf{k}, \boldsymbol{\gamma})\boldsymbol{\gamma},$$
$$2W = \mathbf{k}^2 - (\mathbf{k}, \boldsymbol{\gamma})^2,\ V \equiv 0$$

We denote $G_\lambda = (J + \lambda E)^{-1}$, $\mathbf{e} = |\mathbf{k}|^{-1}\mathbf{k}$.

Thanks to the symmetry of the system with respect to rotations about $\boldsymbol{\gamma}$, we may assume that M is the Poisson sphere. Let u, v be the coordinates on the Poisson sphere defined by the conditions $\rho(-u) = \rho(-v) = 0$, where $\rho(w) = (G_w\boldsymbol{\gamma}, \boldsymbol{\gamma})$, taking values $J_1 \leqslant u \leqslant J_2 \leqslant v \leqslant J_3$. (In these coordinates the problem is integrable when $k = 0$ /11/). The coordinate lines point in the directions of the eigenfunctions of Γ, the eigenvalues are u, v.

The points $\{\boldsymbol{\gamma} : \mathbf{a}_i(\boldsymbol{\gamma}) = 0\}$ are determined by the conditions $(G_{-u}\mathbf{k}, \boldsymbol{\gamma}) = 0$ and $(G_{-v}\mathbf{k}, \boldsymbol{\gamma}) = 0$ and are described geometrically as the points at which the coordinate lines touch circles passing through $\pm\mathbf{e}$.

The critical vectors $\boldsymbol{\omega}_\lambda$ and functions Φ_λ are

$$\boldsymbol{\omega}_\lambda = G_\lambda(\vartheta\boldsymbol{\gamma} - \mathbf{k}),\ \vartheta = (G_\lambda\mathbf{k}, \boldsymbol{\gamma})(G_\lambda\boldsymbol{\gamma}, \boldsymbol{\gamma})^{-1} \tag{5.1}$$
$$\Phi_\lambda = {}^1/_2\lambda[(G_\lambda\mathbf{k}, \mathbf{k}) - (G_\lambda\mathbf{k}, \boldsymbol{\gamma})^2(G_\lambda\boldsymbol{\gamma}, \boldsymbol{\gamma})^{-1}]$$

The critical points of Φ_λ are $\pm\mathbf{e}$ and the points of the circle $L_\lambda = \{(G_\lambda\mathbf{k}, \boldsymbol{\gamma}) = 0\}$. For every $\lambda \neq -J_i$ we obtain a steady motion around L_λ with velocity $\boldsymbol{\omega} = -G_\lambda\mathbf{k}$, corresponding to which are the following values of the integrals:

$$h(\lambda) = {}^1/_2(JG_\lambda^2\mathbf{k}, \mathbf{k}),\ f(\lambda) = {}^1/_2\lambda^2(G_\lambda\mathbf{k})^2 \tag{5.2}$$

When $\lambda = 0$ the function Φ_λ vanishes identically. We obtain a family of steady motions at the velocity ω_0 defined by taking $\lambda = 0$ in (5.1). The values of the integrals are:

$$h = {}^1/_2[(J^{-1}\mathbf{k}, \mathbf{k}) - (J^{-1}\mathbf{k}, \boldsymbol{\gamma})^2(J^{-1}\boldsymbol{\gamma}, \boldsymbol{\gamma})^{-1}],\ f = 0 \tag{5.3}$$

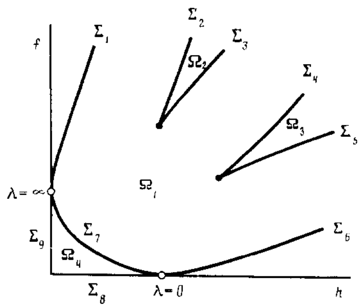
The first of these equalities determines the trajectories: a pair of curves L_h corresponding to the given h. The values corresponding to equilibrium, which is possible at any point because $V \equiv 0$, are:

$$h = 0,\ f = {}^1/_2[\mathbf{k}^2 - (\mathbf{k}, \boldsymbol{\gamma})^2] \tag{5.4}$$

The sets $\{\mu_i = \mathrm{const}\} \cap \{\mathbf{a}_i = 0\}$ consist of isolated points on the Poisson sphere, and since $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0$, there exist no steady motions other than those just described.

The bifurcation set Σ is shown in the figure. The segments Σ_i, $i = 1, \ldots, 7$, are described parametrically by Eqs.(5.2), with λ varying in the following intervals, respectively: $(-\infty, -J_1)$, $(-J_1, -u_0]$, $[-u_0, -J_2)$, $(-J_2, -v_0]$, $[-v_0, -J_3])$, $(-J_3, 0]$, $[0, +\infty)$; u_0, v_0 are the coordinates of the points $\pm\mathbf{e}$. The segments Σ_8, Σ_9 are defined by (5.3) and (5.4), respectively.

Let us determine the type of integral manifolds I_{hf} for the different domains $R^2 \setminus \Sigma$, considering near-critical values of h, f. Suppose that a point of Σ_6 is determined by a parameter value $\lambda = \alpha$. The corresponding critical integral surface contains a steady motion around the circle L_α, with the velocity vector $\boldsymbol{\omega}_\alpha$ a minimum point of the integral F on the sphere $\{\boldsymbol{\omega}: H(\boldsymbol{\omega}) = h\}$. The minimum value of the function Φ_α on L_α is $\min \Phi_\alpha = \alpha h + f$, and therefore all other points lie in the domain $\{\Phi_\alpha - \alpha h > f\} = C_1(\alpha)$ and by (3.2) $M_{hf} = L_\alpha$. If f is decreased we obtain $C_1(\alpha) = M$, i.e., $M_{hf} = \varnothing$, $I_{hf} = \varnothing$. For a slight increase of f, all $C_0(\lambda)$ are empty, all the non-empty $C_1(\lambda)$ constitute a pair of open discs, each of which



contains exactly one of the points $\pm e$ and does not cut L_α; consequently, M_{hf} is an annulus about L_α. The set of possible velocities over the interior points of M_{hf} is a pair of vectors near the minimum of F on the circle $\{H(\omega)=h\}$; over the boundary points it consists of a single vector - the minimum. Thus, for $(h, f) \in \Omega_1$ (see the figure) the integral manifold I_{hf} is a torus.

Similar arguments show that corresponding to a point on Σ_5 we have a steady motion which, on passing to Ω_2, produces another torus. When we cross Σ_4 into Ω_1, the two tori merge along the points of the steady motion to produce a single torus. Upon passage through Σ_3 and Σ_2 the evolution takes place in the opposite sense: the torus splits into two, one of which then contracts to an isolated trajectory of steady motion and disappears. The other torus contracts to a steady trajectory over Σ_1. If $(h, f) \in \Sigma_8$, then I_{hf} is a pair of steady trajectories over the curves L_h. A slight increase in f produces a torus around each of these trajectories; passage through Σ_7 into Ω_1 combines the two tori into one. Thus, the integral manifolds corresponding to the points of $\Omega_{2,3,4}$, are pairs of tori. For points outside Ω_i, Σ_j they are empty. The description of the DPMs M_{hf} for all values of h, f involves distinguishing a large number of different cases, depending on the values of $\mathbf{k}, J_i$, and is thus extremely complicated.

We may conclude from the above analysis that small perturbations of the steady trajectories corresponding to points of Σ_i, $i = 1, 2, 5, 6, 8$, produces motions confined to tori in their neighbourhoods; hence these trajectories are stable with respect to some of the variables. Stable equilibria are obtained at the points $\pm e$ if $f = 0$; small perturbations produce DPMs which are small domains in the neighbourhoods of these points.

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